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Note

Tales of Hoffman: Three extensions of Hoffman's bound on the graph chromatic number ☆

Yonatan Bilu

Department of Molecular Genetics, Weizmann Institute of Science, Rehovot 76100, Israel

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Abstract

Hoffman's bound on the chromatic number of a graph states that $\chi \geqslant 1 - \lambda_1/\lambda_n$. Here we show that the same bound, or slight modifications of it, hold for several graph parameters related to the chromatic number: the vector coloring number, the ψ-covering number and the λ-clustering number.

Keywords: Graph chromatic number; Vector chromatic number; Hoffman's bound; Eigenvalues; Positive semidefinite matrices

1. Introduction

Let G be a graph on n vertices, χ its chromatic number, and A its adjacency matrix. Let λ_1 and λ_n be the largest and least eigenvalues of A. Hoffman's bound [2] states that:

$$\chi \geqslant 1 - \frac{\lambda_1}{\lambda_n}.$$

In fact, Hoffman's proof gives a stronger result. let W be any *weighted adjacency matrix* of G, that is, a non-negative symmetric matrix s.t. $W_{i,j} = 0$ iff (i, j) is not an edge in G, then the same bound holds with λ_1 and λ_n being the largest and least eigenvalues of W.

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E-mail address: yonatan.bilu@weizmann.ac.il.

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Karger, Motwani and Sudan [3] define a quadratic programming relaxation of the chromatic number, called the *vector chromatic number*. This is the minimal k such that there exist unit vectors $u_1, \ldots, u_n \in \mathbb{R}^n$ with

$$\langle u_i, u_j \rangle \leqslant -\frac{1}{k-1},$$

whenever (i, j) is an edge in the graph.

Let χ_v denote the vector chromatic number of G. Karger, Motwani and Sudan observe that $\chi_v \leqslant \chi$. In this note we show that Hoffman's bound holds for this parameter as well:

Theorem 1. *Let G be a graph, and W a weighted adjacency matrix for it. Let λ_1 and λ_n be the largest and least eigenvalues of W. Then*

$$\chi_v(G) \geqslant 1 - \frac{\lambda_1}{\lambda_n}.$$

Let ψ be a real-valued function on graphs, such that $\psi(G) = 1$ if G is edgeless, and $\psi(G) \geqslant \chi(G)$ for all G. The ψ*-covering* number of a graph G is the smallest integer k such that there exist subsets $S_1, \ldots, S_k$ of $V(G)$ with the property that

$$\sum_{i:\ v \in S_i} \frac{1}{\psi(S_i)} \geqslant 1.$$

(Note that there is a slight abuse of notation here—by $\psi(S_j)$ we refer here and in the sequel to $\psi(G[S_j])$.)

Amit, Linial and Matoušek, who introduced this notion in [1], show that the ψ*-covering* number of a graph G is bounded between $\sqrt{\chi(G)}$ and $\chi(G)$. They ask whether better lower bounds can be proven for the special cases $\psi(G) = \Delta(G) + 1$ and $\psi(G) = \mathrm{dgn}(G) + 1$. Here $\Delta(G)$ denotes the maximal degree in G, and $\mathrm{dgn}(G)$ the degeneracy of a graph G—the maximum over all induced subgraphs H of G of $\delta(H)$, the minimal vertex degree in H.

To state our result, we will need a couple of ad-hoc definitions:

A graph G has a *c-vertex cover* if there exists a cover $E(G) = \bigcup_{i \in V(G)} E_i$ such that for all $i \in V(G)$, $E_i \subset \{e \in E:\ i \in e\}$, and $|E_i| \leqslant c$. Denote by $L_{\psi,\alpha}(G)$ the minimal k such that there exist k subsets of G, $S_1, \ldots, S_k$, so that for all $i = 1, \ldots, k$, $G[S_i]$ has a $\alpha \cdot \psi(G)$-vertex cover, and for every $v \in G$, $\sum_{i:\ v \in S_i} 1/\psi(S_i) \geqslant 1$.

Observe that all graphs have a $\psi(G)$-vertex cover for $\psi(G) = \mathrm{dgn}(G) + 1$, and a $\frac{1}{2}\psi(G)$-vertex cover when $\psi(G) = \Delta(G) + 1$. So in these cases, $L_{\psi,1}$ and $L_{\psi,1/2}$, respectively, are exactly the ψ-covering numbers. Note also that $\alpha = 0$ means that the S_i are independent sets. Thus, since $\psi = 1$ on such sets, $L_{\psi,0} = \chi$.

Theorem 2. *Let G be a graph with average degree d, and let λ_n be the least eigenvalue of its adjacency matrix. Then*

$$L_{\psi,\alpha}(G) \geqslant \frac{d - \lambda_n}{2\alpha - \lambda_n}.$$

Note that when the graph is regular and $\alpha = 0$, this is the same as Hoffman's bound. For random d-regular graphs, $|\lambda_n| = O(\sqrt{d})$ and $\chi = \Theta(d/\log d)$. So in this case (if α is taken small) the bound is slightly better than $\sqrt{\chi}$ mentioned above.

Finally, we are interested in a graph parameter that has to do with how well a graph can be partitioned into sparse clusters.

Let W be a weighted adjacency matrix of a graph G. A partition $V = \dot{\bigcup}_{i=1}^{k} C_i$ is a *λ-clustering* of G into k clusters if

$$\max_{i \in [k]} \lambda_1(C_i) \leqslant \lambda,$$

where $\lambda_1(C_i)$ is the largest eigenvalue of the (weighted) subgraph spanned by the vertices in C_i. The *λ-clustering number* of G is the minimal k such that there exists a λ-clustering of G into k clusters.

It is not hard to see that the 0-clustering number is identical to the chromatic number. We show that Hoffman's bound can also be extended to this graph parameter.

Theorem 3. *Let G be a graph, and W a weighted adjacency matrix for it. Let λ_1 and λ_n be the largest and least eigenvalues of W. The λ-clustering number of the graph is at least*:

$$\frac{\lambda_1 - \lambda_n}{\lambda - \lambda_n}.$$

2. Vectorial characterization of the least eigenvalue

All three results mentioned in the previous section rely on the following observation:

Lemma 4. *Let A be a real symmetric matrix and λ_n its least eigenvalue*:

$$\lambda_n = \min \frac{\sum_{i,j=1}^{n} A_{i,j} \langle v_i, v_j \rangle}{\sum_{i=1}^{n} \|v_i\|_2^2}, \tag{1}$$

where the minimum is taken over all $v_1, \ldots, v_n \in \mathbb{R}^n$.

Proof. By the Rayleigh–Ritz characterization, λ_n equals:

$$\begin{aligned} \min\ & \sum_{i,j} A_{i,j} x_i x_j \\ \text{s.t.}\ & x \in \mathbb{R}^n, \\ & \|x\|_2 = 1. \end{aligned}$$

Denote by PSD_n the cone of $n \times n$ positive semi-definite matrices. For each unit vector $x \in \mathbb{R}^n$, let X be the matrix $X_{i,j} = x_i x_j$. This is a positive semi-definite matrix of rank 1 and trace 1, and all such matrices are obtained in this way. Hence, λ_n equals:

$$\begin{aligned} \min\ & \sum_{i,j} A_{i,j} X_{i,j} \\ \text{s.t.}\ & X \in \mathrm{PSD}_n, \\ & \mathrm{rank}(X) = 1, \\ & \mathrm{tr}(X) = 1. \end{aligned}$$

However, the rank restriction is superfluous. It restricts the solution to an extreme ray of the cone PSD_n, but, by convexity, the optimum is attained on an extreme ray anyway. Hence, λ_n equals:

$$\begin{aligned}
&\min \sum_{i,j} A_{i,j} X_{i,j} \\
&\text{s.t.} \quad X \in \mathrm{PSD}_n, \\
&\qquad \mathrm{tr}(X) = 1.
\end{aligned}$$

Now, think of each $X \in \mathrm{PSD}_n$ as a Gram matrix of n vectors, $v_1, \ldots, v_n$ (i.e., $X_{i,j} = \langle v_i, v_j \rangle$). An equivalent formulation of the above is thus:

$$\begin{aligned}
&\min \sum_{i,j} A_{i,j} \langle v_i, v_j \rangle \\
&\text{s.t.} \quad v_i \in \mathbb{R}^n \quad \text{for } i = 1, \ldots, n, \\
&\qquad \sum_{i=1}^{n} \|v_i\|_2^2 = 1.
\end{aligned}$$

Clearly, this is equivalent to (1). □

3. Proofs of theorems

Proof of Theorem 1. Let G be a graph on n vertices with vector chromatic number χ_v. Let $W \neq 0$ be a symmetric matrix such that $W_{i,j} = 0$ whenever $(i, j) \notin E$. Let λ_1 and λ_n be the largest and least eigenvalues of W.

We choose vectors $v_1, \ldots, v_n$, and look at the bound they give on λ_n in Lemma 4. Let $u_1, \ldots, u_n \in S^n$ be vectors on which the vector chromatic number is attained. That is, $\langle u_i, u_j \rangle \leqslant -1/(\chi_v - 1)$ for $(i, j) \in E$, and $\|u_i\|_2 = 1$. Let $\alpha \in \mathbb{R}^n$ be an eigenvector of W corresponding to λ_1. Set $v_i = \alpha_i \cdot u_i$.

Since $W_{i,j} = 0$ whenever $\langle u_i, u_j \rangle > -1/(\chi_v - 1)$, by Lemma 4,

$$\begin{aligned}
\lambda_n &\leqslant \frac{\sum_{i,j} W_{i,j} \alpha_i \alpha_j \langle u_i, u_j \rangle}{\sum_{i=1}^{n} \alpha_i^2 \cdot \|u_i\|_2^2} \leqslant -\frac{1}{\chi_v - 1} \cdot \frac{\sum_{i,j} W_{i,j} \alpha_i \alpha_j}{\sum_i \alpha_i^2} \\
&= -\frac{1}{\chi_v - 1} \cdot \frac{\alpha^t W \alpha}{\|\alpha\|^2} = -\frac{1}{\chi_v - 1} \cdot \lambda_1.
\end{aligned}$$

Equivalently, $\chi_v \geqslant 1 - \lambda_1/\lambda_n$, as claimed. □

Proof of Theorem 2. Denote $k = L_{\psi,\alpha}(G)$, and let $u_1, \ldots, u_k$ be the vertices of the regular $(k-1)$-dimensional simplex centered at 0—i.e., $\langle u_i, u_j \rangle = 1$ when $i = j$ and $-1/(k-1)$ otherwise. Again we choose vectors $v_1, \ldots, v_n$. We do so probabilistically. Let $S_1, \ldots, S_k$ be the subsets attaining the value k. For each i, v_i will be chosen from among the u_j's such that $i \in S_j$. Specifically, let $h_i = \sum_{j:\, i \in S_j} 1/\psi(S_j)$. Let $p_{i,j} = h_i^{-1}(1/\psi(S_j))$ be the probability that v_i is chosen to be u_j. Note that $h_i \geqslant 1$, and so $p_{i,j} \leqslant 1/\psi(S_j)$.

Say that an edge is "bad" if both its endpoints are assigned the same vector. For a given j, the probability that an edge $(i, i') \in E(S_j)$ is "bad" because both endpoints were assigned to u_j is $p_{i,j} p_{i',j}$. Thus, the expected number of "bad" edges is at most:

$$\sum_{j=1}^{k} \sum_{(i,i') \in E(S_j)} p_{i,j} p_{i',j}.$$

Each S_j has a $\alpha \cdot \psi(G)$-vertex cover $E(S_j) = \bigcup_{i \in S_j} E_{j,i}$. Summing the expression above according to this cover (some edges might be counted more than once) we get that the expected number of "bad" edges is at most:

$$\sum_{j=1}^{k} \sum_{i \in S_j} \sum_{i' \colon (i,i') \in E_{j,i}} p_{i,j} p_{i',j} \leqslant \sum_{j=1}^{k} \sum_{i \in S_j} p_{i,j} |E_{j,i}| \frac{1}{\psi(S_j)}$$
$$\leqslant \sum_{j=1}^{k} \sum_{i \in S_j} p_{i,j} \alpha = \alpha \sum_{i \in G} \sum_{j \colon i \in S_j} p_{i,j} = \alpha n.$$

In particular, there is a choice of v_i's such that the number of "bad" edges is at most αn. Assume this is the case. If $(i,j) \in E(G)$ is a "bad" edge then $\langle v_i, v_j \rangle = 1$. Otherwise $\langle v_i, v_j \rangle = -1/(k-1)$.

Lemma 4 now gives:

$$\lambda_n \leqslant \left(2\alpha n - \frac{1}{k-1}(dn - 2\alpha n)\right) \Big/ n = \frac{2k\alpha}{k-1} - \frac{d}{k-1},$$

or $(k-1)\lambda_n \leqslant 2k\alpha - d$. Equivalently, $k \geqslant (d - \lambda_n)/(2\alpha - \lambda_n)$. □

The following variation of Theorem 2, which does not make use of the c-covering number, can be proven in essentially the same way. Let G be a graph with average degree d, and let λ_n be the least eigenvalue of its adjacency matrix. Let $L'_{\psi,\alpha}$ be the minimal k such that there exist k subsets of G, $S_1, \ldots, S_k$, so that for all $i = 1, \ldots, k$, $|E(S_i)| \leqslant \alpha \cdot \psi(S_i) \cdot |V(S_i)|$, and for every $v \in G$, $\sum_{i \colon v \in S_i} 1/\psi(S_i) = 1$. Then

$$L'_{\psi,\alpha} \geqslant \frac{d - \lambda_n}{2\alpha - \lambda_n}.$$

Proof of Theorem 3. Denote the λ-clustering number of W by k. Let $u_1, \ldots, u_k \in R^n$ be the vertices of a regular simplex centered at the origin, as above. Let $\alpha \in \mathbb{R}^n$ be an eigenvector of W, corresponding to λ_1. Let $C_1, \ldots, C_k$ be a λ-clustering of G. Define $\phi : V \to [k]$ to be the index of the cluster containing a vertex. That is, for each $i \in V$, $i \in C_{\phi(i)}$. Define W_l to be the weighted $n \times n$ adjacency matrix of the edges of the subgraph spanned by C_l. In other words, W_l is identical to W on edges with both vertices in C_l, and 0 otherwise. Note that $\lambda_1(W_l) \leqslant \lambda$. Set $v_i = \alpha_i \cdot u_{\phi(i)}$.

By Lemma 4,

$$\lambda_n \leqslant \frac{\sum_{i,j} \alpha_i \alpha_j \langle u_{\phi(j)}, u_{\phi(j)} \rangle W_{i,j}}{\sum_{i=1}^{n} \alpha_i^2 \cdot \|u_{\phi(i)}\|_2^2}$$
$$= -\frac{1}{k-1} \cdot \frac{\sum_{i,j \colon \phi(i) \neq \phi(j)} \alpha_i \alpha_j W_{i,j}}{\sum_i \alpha_i^2} + \frac{\sum_{i,j \colon \phi(i) = \phi(j)} \alpha_i \alpha_j W_{i,j}}{\sum_i \alpha_i^2}$$
$$= -\frac{1}{k-1} \cdot \frac{\alpha^t W \alpha}{\|\alpha\|^2} + \frac{k}{k-1} \frac{\sum_{l=1}^{k} \sum_{i,j \in C_l} \alpha^t W_l \alpha}{\sum_i \alpha_i^2}$$
$$\leqslant -\frac{1}{k-1}\lambda_1 + \frac{k}{k-1}\lambda.$$

Equivalently, $k \geqslant (\lambda_n - \lambda_1)/(\lambda_n - \lambda)$. □

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